Equivalence of Riesz and Baez-Duarte criterion for the Riemann Hypothesis

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Abstract

We investigate the relation between the Riesz and the Baez-Duarte criterion for the Riemann Hypothesis. In particular we present the relation between the function $R(x)$ appearing in the Riesz criterion and the sequence c_k appearing in the Baez-Duarte formulation. It is shown that $R(x)$ can expressed by c_k and vice versa the sequence c_k can be obtained from the values of $R(x)$ at integer arguments. We give also some relations involving c_k and $R(x)$, in particular value of the alternating sum of c_k.

1. Introduction.

The Riemann Hypothesis (RH) states that the nontrivial zeros of the function:

$$\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \tag{1}$$

where $\Re(s) > 0$ and $s \neq 1$, are simply and have the real part equal to half, i.e. $\Re(s) = \frac{1}{2}$. There are probably over 100 statements equivalent to RH, see eg. [1], [2], [3]. In the beginning of XX century M. Riesz [4] has considered the function:

$$R(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{(k-1)!\zeta(2k)} = x \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!\zeta(2k+2)}. \tag{2}$$

Unconditionally it can be proved that $R(x) = \mathcal{O}(x^{1/2+\epsilon})$, see [1] §14.32. Riesz has proved that the Riemann Hypothesis is equivalent to slower increasing of the function $R(x)$:

$$RH \Leftrightarrow R(x) = \mathcal{O}\left(x^{1/4+\epsilon}\right). \tag{3}$$

A few years ago L. Baez-Duarte [5] [6] considered the sequence of numbers c_k defined by:

$$c_k = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{1}{\zeta(2j+2)}. \tag{4}$$

He proved that RH is equivalent to the following rate of decreasing to zero of the above sequence:

$$RH \Leftrightarrow c_k = \mathcal{O}(k^{-\frac{3}{4}+\epsilon}) \qquad \text{for each} \;\; \epsilon > 0. \tag{5}$$

Furthermore, if ϵ can be put zero, i.e. if $c_k = \mathcal{O}(k^{-\frac{3}{4}})$, then the zeros of $\zeta(s)$ are simply. Baez-Duarte also proved in [6] that it is not possible to replace $\frac{3}{4}$ by larger exponent. Although the title of the Baez-Duarte paper was *A sequential Riesz-like criterion for the Riemann Hypothesis* he has not pursued further relation between c_k and $R(x)$.

In this paper we are going to establish the relation between c_k and $R(x)$. In Sect. 2 we will present formulae allowing to obtain values of $R(x)$ and c_k much faster than from (2) and (4). In Sect. 3 we will use the fact that c_k can be obtained as forward differences of a appropriate sequence to express $R(x)$ in terms of c_k. Next we will prove equivalence of the Riesz and Baez-Duarte criterion for RH. In the mathematical logic the *iff* obeys the transitivity rule:

$$(p \Leftrightarrow q \;\; \text{AND} \;\; q \Leftrightarrow s) \Rightarrow (p \Leftrightarrow s)$$

thus from (3) and (5) we have that $R(x) = \mathcal{O}\left(x^{1/4+\epsilon}\right) \Leftrightarrow c_k = \mathcal{O}(k^{-\frac{3}{4}+\epsilon})$.

However we will prove equivalence (Riesz criterion) $\Leftrightarrow$ (Baez-Duarte criterion) in a more general form, namely the exponents 1/4 and 3/4 will be replaced by arbitrary parameter δ and combination $1-\delta$: $c_k = \mathcal{O}\left(k^{-\delta}\right) \Leftrightarrow R(x) = \mathcal{O}\left(x^{1-\delta}\right)$. In the final Section we will speculate on some equations involving c_k and $R(x)$, in particular we will calculate the alternating sum $\sum_{k=0}^{\infty}(-1)^k c_k$.

2. Some facts on the $R(x)$ and c_k

The most comprehensive source of information about the Riesz function $R(x)$ we have found on the Wikipedia [9]. For large negative x function $R(x)$ tends to xe^{-x}. For positive x the behaviour of $R(x)$ is much more difficult to reveal because the series (2) is very slowly convergent. Applying Kummer's acceleration convergence method gives

$$R(x) = x \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} \, e^{-\frac{x}{n^2}} \tag{6}$$

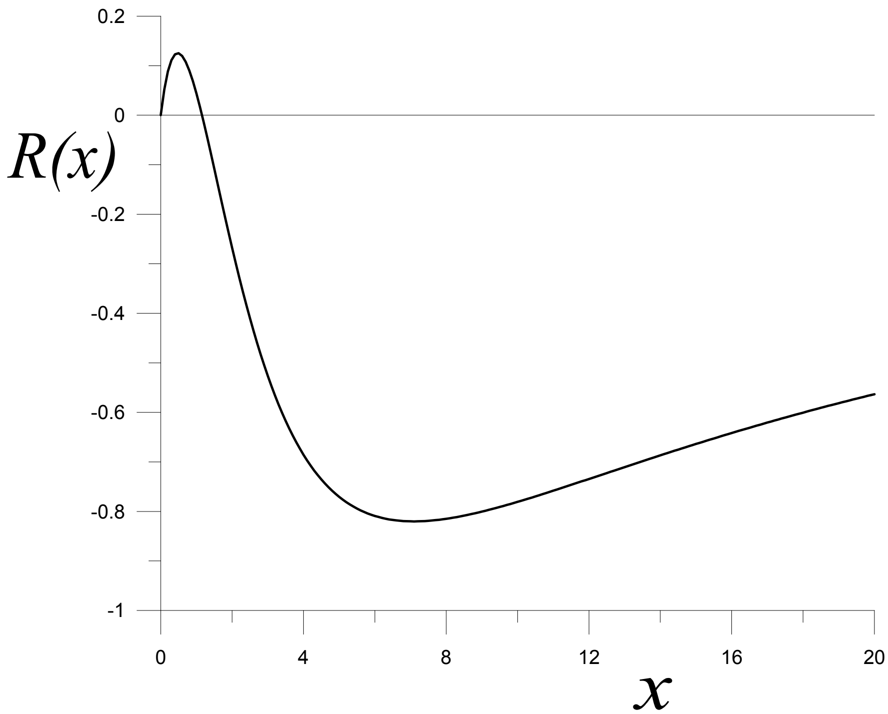

Fig.1 The plot of $R(x)$ for $x \in (1, 20)$. Such a short interval is chosen to show the first zero of $R(x)$.

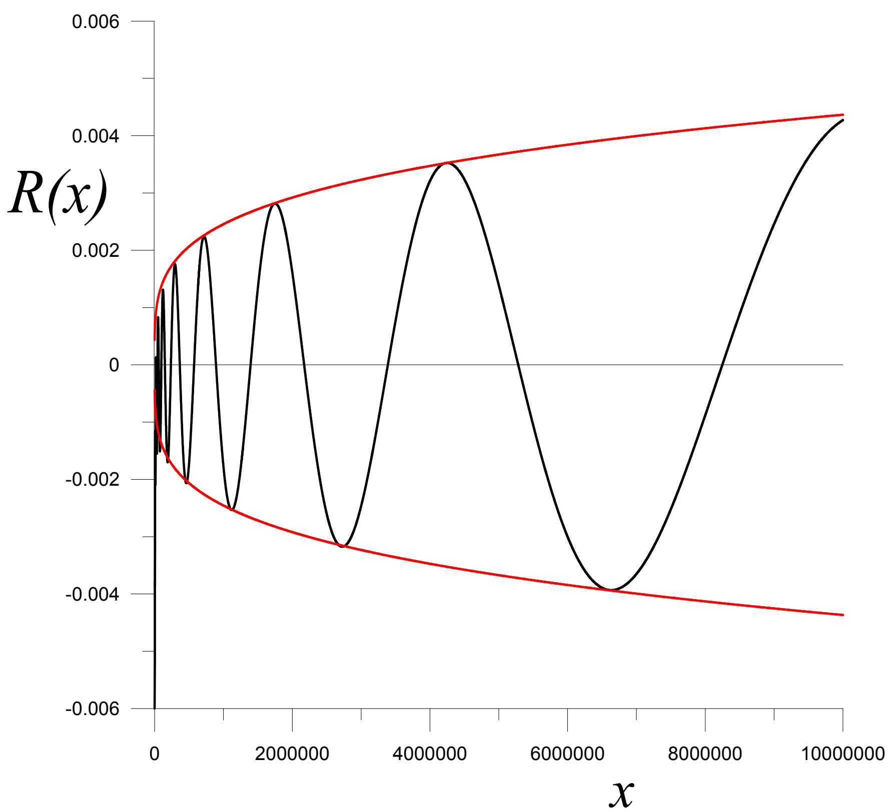

Fig.2 The plot of $R(x)$ for $x \in (0, 10^7)$. The part of $R(x)$ smaller than -0.006 is skipped.

where μ is the Möbius function:

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n \text{ is divisible by a square of a prime} \\ (-1)^k & \text{if } n \text{ is a product of } k \text{ different primes} \end{cases} \tag{7}$$

Repeating Kummer's procedure gives:

$$R(x) = x\left(\frac{6}{\pi^2} + \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2}\left(e^{-\frac{x}{n^2}} - 1\right)\right). \tag{8}$$

Using this formula we were able to produce the plot of $R(x)$ for x up to 10^7, see Fig.1 and Fig.2. The first nontrivial zero of $R(x)$ is $x_0 = 1.1567116438\ldots$. The envelops on the Fig.2 (in red) are given by the equations

$$y(x) = \pm A x^{\frac{1}{4}}, \tag{9}$$

where $A = 0.777506\ldots \times 10^{-5}$.

It is very time consuming to calculate values of the sequence c_k directly from the definition (4), see [7], [8]. The point is, that for large j $\zeta(2j)$ is practically 1, and to distinguish it from 1 high precision calculations are needed. The experience of [8] showed that to calculate c_k from (4) roughly $k\log_{10}(k)$ digits accuracy is needed. However in [6] Baez-Duarte gave the explicit formula[1] for c_k valid for large k:

$$c_{k-1} = \frac{1}{2k}\sum_{\rho} \frac{k^{\frac{\rho}{2}}\Gamma(1-\frac{\rho}{2})}{\zeta'(\rho)} + o(1/k) \tag{10}$$

where the sum runs for nontrivial zeros ρ of $\zeta(s)$: $\zeta(\rho) = 0$ and $\Im(\rho) \neq 0$. Maślanka in [7] gives the similar formula which contains the term hidden in o(1/k) in (10). Let us introduce the notation

$$\frac{\Gamma(1-\frac{\rho_i}{2})}{\zeta'(\rho_i)} = a(\rho_i) + ib(\rho_i) \equiv a_i + ib_i. \tag{11}$$

Assuming $\rho_i = \frac{1}{2} + i\gamma_i$ it can be shown [8] that a_i and b_i very quickly decrease to zero:

$$\left|\frac{\Gamma(1-\frac{\rho_i}{2})}{\zeta'(\rho_i)}\right| \sim e^{-\pi\gamma_i/4} \tag{12}$$

Finally we obtain for large k:

$$c_{k-1} = \frac{1}{k^{\frac{3}{4}}}\sum_{i=1}^{\infty}\left\{a_i \cos\left(\frac{\gamma_i \log(k)}{2}\right) - b_i \sin\left(\frac{\gamma_i \log(k)}{2}\right)\right\}. \tag{13}$$

[1] There is an error in [6] and there should be no minus sign in front of c_{k-1} in formulae (1.11), (1.12), (4.1), (4.11) in [6].

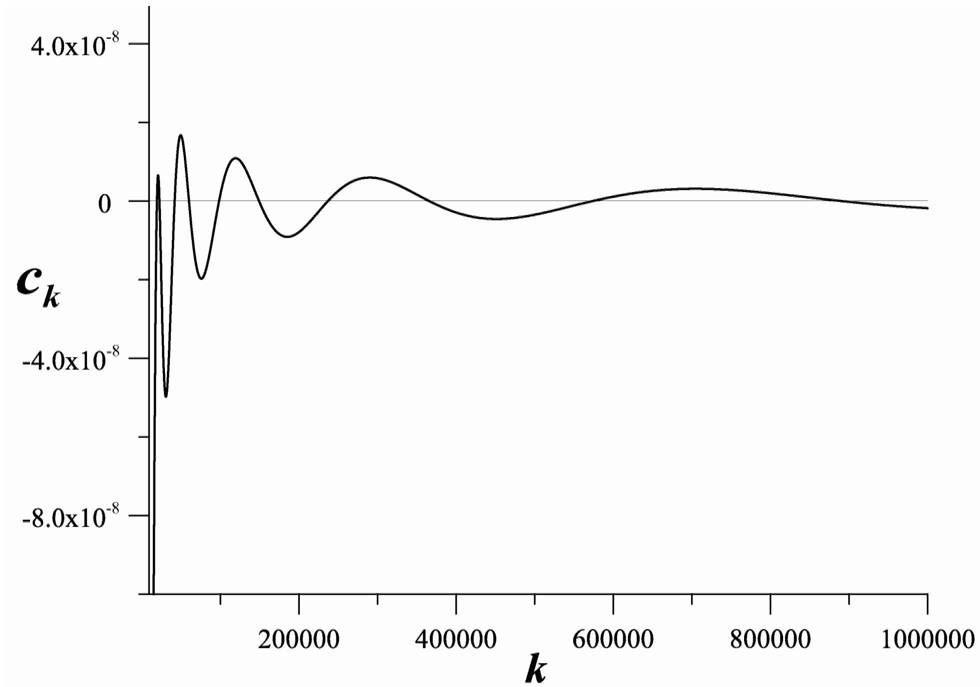

Fig.3 The plot of c_k for $k \in (1, 10^6)$.

The above formula explains oscillations seen on the plots of c_k published in [6] and [7], see Fig.3 . Because these curves are perfect cosine–like graphs on the plots versus $\log(k)$ it means that in fact in the above formula (13) it suffices to maintain only the first zero and skip all remaining terms in the sum.

3. Relation between $R(x)$ and c_k

The values of c_k can be obtained as the first elements of the sequence of forward differences of the sequence:

$$f_0^0 = \frac{1}{\zeta(2)} \qquad f_1^0 = \frac{1}{\zeta(4)} \qquad f_2^0 = \frac{1}{\zeta(6)} \qquad f_3^0 = \frac{1}{\zeta(8)} \qquad f_4^0 = \frac{1}{\zeta(10)} \qquad \cdots \tag{14}$$

Then we form forward differences:

$$f_l^k = f_l^{k-1} - f_{l+1}^{k-1} \tag{15}$$

and we have that $c_k = f_0^k$. We will recall some facts from finite difference calculus adapted for our purposes [10]: Let us define as usual the shift operator E:

$$Ef(k) = f(k+1).$$

Next we introduce sequence:

$$c_k = (1-E)^k f(0) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j f(j). \tag{16}$$

Then the following equalities holds [10]:

$$e^{x(1-E)} f(0) = \sum_{k=0}^{\infty} \frac{c_k}{k!} x^k,$$

$$e^{x(1-E)} f(0) = e^x e^{-xE} f(0) = e^x \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} f(k),$$

from which it follows that:

$$\sum_{k=0}^{\infty} \frac{c_k}{k!} x^k = e^x \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} f(k).$$

In our case we put

$$f(k) = \frac{1}{\xi(2k+2)}$$

and finally we have:

$$\sum_{k=0}^{\infty} \frac{c_k}{k!} x^k = e^x \sum_{k=0}^{\infty} \frac{(-x)^k}{k! \xi(2k+2)} = \frac{e^x}{x} R(x). \tag{17}$$

Thus $R(x)$ can be reconstructed from c_k. Vice versa, we will see later, see (26), that within some accuracy c_k can be obtained from $R(x)$. In the paper [8] it was suggested that the duality holds:

$$c_k = \mathcal{O}\left(k^{-\delta}\right) \Leftrightarrow R(x) = \mathcal{O}\left(x^{1-\delta}\right). \tag{18}$$

Putting $\delta = \frac{3}{4} - \epsilon$ gives original criteria (3) and (5). In fact we will prove it in the following form:

Theorem 1. *The sequence c_k defined by (4) decrease like $c_k = \mathcal{O}\left(k^{-\delta}\right)$ if and only if the function $R(x)$ defined by (2) grows like $R(x) = \mathcal{O}\left(x^{1-\delta}\right)$, where $\delta < 3/2$.*

Remark: In fact δ is smaller than 3/4, as shown by Baez-Duarte in [6].

Proof: The reasoning that if $c_k = \mathcal{O}\left(k^{-\delta}\right)$ then $R(x) = \mathcal{O}\left(x^{1-\delta}\right)$ we will base on the following facts from Exercises 67 – 71 in the Part IV of famous book of G. Polya and G. Szegö [11]. We summarize these facts adapted for our purposes in the form: Let

$$f(x) = \sum_{k=0}^{\infty} a_k x^k, \tag{19}$$

where a_k are positive and decrease monotonically $a_0 \geq a_1 \geq a_2 \ldots \geq a_k \geq \ldots$. Let α be defined by

$$\log a_k \sim -\frac{k \log k}{\alpha} \tag{20}$$

and next the parameter b be determined from

$$\log f(x) \sim bx^{\alpha}. \tag{21}$$

Then for large x the following asymptotic relation is fulfilled:

$$\sum_{k=1}^{\infty} k^{\delta} a_k x^k \sim (\alpha b x^{\alpha})^{\delta} f(x). \tag{22}$$

In our case we have $a_k = 1/k!$ thus $f(x) = e^x$ and from Stirling formula we have $\alpha = 1$ and next $b = 1$ and hence we have from above formula for large x:

$$\sum_{k=1}^{\infty} \frac{k^{-\delta} x^k}{k!} \sim x^{-\delta} e^x. \tag{23}$$

If we assume that $|c_k| < Ak^{-\delta}$ then we have

$$\sum_{k=1}^{\infty} \frac{|c_k| x^k}{k!} < Ax^{-\delta} e^x \tag{24}$$

and from (17) it follows:

$$|R(x)| < Ax^{1-\delta} \tag{25}$$

what is a desired inequality.

We will show now the opposite implication: from $R(x) = \mathcal{O}\left(x^{1-\delta}\right)$ it follows that $c_k = \mathcal{O}\left(k^{-\delta}\right)$. In Appendix we prove the following inequality:

$$\left|\frac{R(k)}{k} - c_k\right| \leq \frac{3\sqrt{\pi}}{16} k^{-3/2} + \mathcal{O}\left(k^{-2}\right) \tag{26}$$

Because $|c_k| - |R(k)/k| < |c_k - R(k)/k|$ and we assume $|R(k)| \leq Bk^{1-\delta}$ thus we have

$$|c_k| \leq Bk^{-\delta} + \mathcal{O}\left(k^{-\frac{3}{2}}\right) \tag{27}$$

To avoid nonsense δ should be smaller than 3/2 and in fact Baez-Duarte showed [6] that existence of zeros on the critical axis requires $\delta < 3/4$.

□

The comparison of the above bound (26) with real computer data is given in the Fig.4. Here the fit (red line) was obtained by the least square method from the data with $k > 10000$ to avoid transient regime and it is given by the equation

$y = 0.0117483x^{-1.52655}$. The fact that approximately $c_k \approx R(k)/k$ was observed previously by S. Beltraminelli and D. Merlini [12]. It can be explained heuristically as follows: Baez-Duarte gives in [6] despite (4) a few formulae for c_k. We need here the following expression being the transformation of (4):

$$c_k = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} \left(1 - \frac{1}{n^2}\right)^k . \tag{28}$$

For large k we can write:

$$c_k = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} \left(1 - \frac{k}{kn^2}\right)^k \approx \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} e^{-n^2/k} \tag{29}$$

and comparing it with (6) we get $c_k \approx R(k)/k$ for large k.

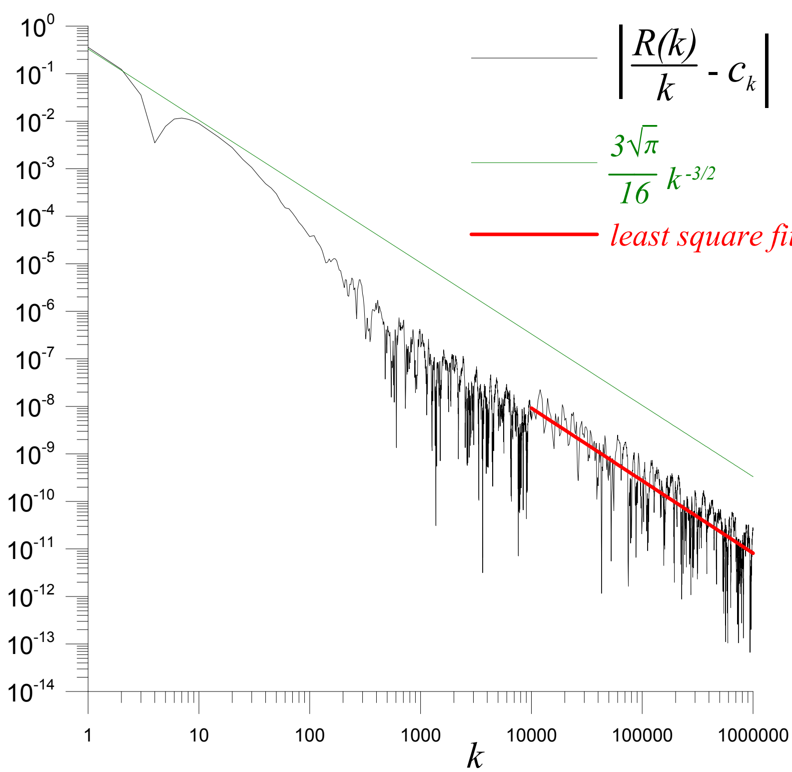

Fig. 4 The log-log plot of $|c_k - R(k)/k|$ for $k \in (0, 10^6)$.

4 Some other relations

Using the formula (28) it is possible to calculate the alternating sum of c_k:

$$\sum_{k=0}^{\infty} (-1)^k c_k = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{1}{\zeta(2k)}. \tag{30}$$

Numerically this sum is $\sum_{k=0}^{\infty}(-1)^k c_k = 0.7825279853253842345766 88\ldots$. This number probably can not be expressed by other known constants, because the Simon Plouffe inverter failed to find any relation [13]. By the Abel's summation the r.h.s. can be written as:

$$\sum_{k=1}^{\infty}\frac{1}{2^k}\frac{1}{\zeta(2k)} = 1 + \int_2^{\infty}\left(1-\frac{1}{2^{\lfloor x/2\rfloor}}\right)\frac{\zeta'(x)}{\zeta^2(x)}dx. \tag{31}$$

In fact more general than (30) formula holds:

$$\sum_{k=0}^{\infty} c_k s^k = \frac{1}{1-s}\sum_{k=0}^{\infty}\left(\frac{-s}{1-s}\right)^k\frac{1}{\zeta(2k+2)}, \tag{32}$$

where $-1 \leq s < \frac{1}{2}$. Here we have made use of the identity

$$\sum_{n=1}^{\infty}\left(1-\frac{1}{n^2}\right)^k s^k = \frac{1}{1-s}\sum_{k=0}^{\infty}\left(\frac{-s}{1-s}\right)^k\frac{1}{n^{2k}}. \tag{33}$$

The l.h.s. is convergent for $-1 \leq s < 1$ while the r.h.s. converges for $-\infty < s < 1/2$.

The question of the convergence of the sum $\sum_{k=0}^{\infty} c_k$ is much more complicated. Formally summing both sides of (16) we get:

$$\sum_{k=0}^{\infty} c_k = E^{-1}f(0) = f(-1) \tag{34}$$

As in our case $f(k) = 1/\zeta(2k+2)$ we have

$$\sum_{k=0}^{\infty} c_k = \frac{1}{\zeta(0)} = -2 \tag{35}$$

because $\zeta(0) = -\frac{1}{2}$, see e.g. [1], p.19. The partial sums $\sum_{k=0}^{n} c_k$ indeed initially tend from above to -2, but for $n \approx 91000$ the partial sum crosses -2 and around $n \approx 100000$ the partial sum starts to increase. These oscillations begins to repeat with growing amplitude around -2. The Fig. 5 shows the plot of distances of the partial sums $\sum_{k=0}^{n} c_k$ from -2. Let us remark that at $n \sim 10^8$ the amplitude is rather very small: of the order 0.001. When we retain in (13) only the first zero γ_1 it can be shown that this amplitude grows like $n^{1/4}$, thus it appears that the above formal derivation (34) is wrong and the sum $\sum_{k=0}^{n} c_k$ is divergent.

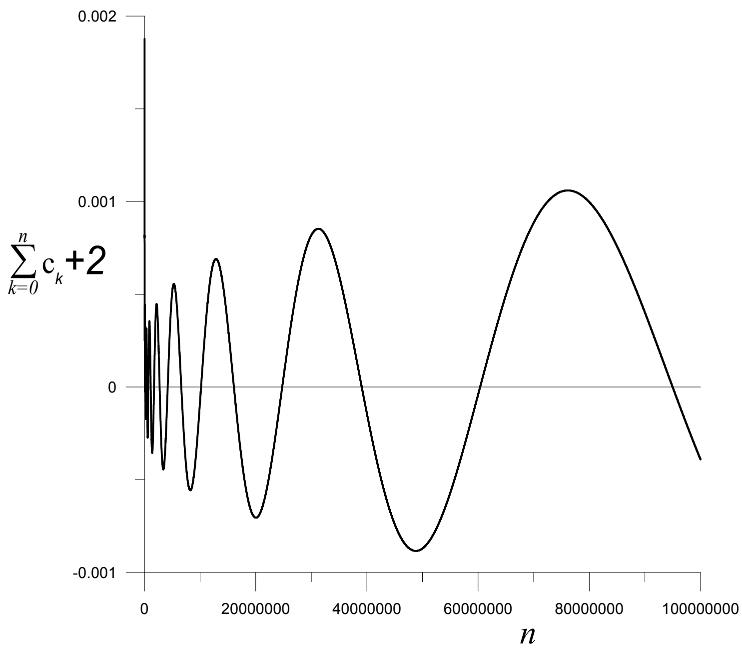

Fig. 5 The distance from -2 of the partial sums $\sum_{k=0}^{n}$ for $n = 1, \ldots 10^8$.

We have made analogous plot of partial sums $\sum_{k=0}^{n}(-1)^k c_k$ and there we have seen oscillations around the limit value $s = 0.782527985\ldots$ of *decreasing* amplitude. Thus we speculate, that this partial sums behave as c_k and $R(x)$ accordingly:

$$\left|\sum_{k=0}^{n}(-1)^k c_k - s\right| = \mathcal{O}\left(n^{-\frac{3}{4}}\right), \tag{36}$$

$$\left|\sum_{k=0}^{n} c_k + 2\right| = \mathcal{O}\left(n^{\frac{1}{4}}\right). \tag{37}$$

Finally we would like to argue in favour of the two strange approximate equalities. Both follows from $c_k \approx R(k)/k$ for large k. The first follows when write this relation with the help of (2) and (4):

$$\sum_{j=0}^{\infty} \frac{(-1)^j k^j}{j!\zeta(2j+2)} \approx \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{1}{\zeta(2j+2)}. \tag{38}$$

On both sides there appears inverses of $\zeta(2n)$. We have checked numerically that the difference between these two sums very quickly tends to zero.

The second formula we get when in (17) we put instead of c_k simply $R(k)/k$:

$$\frac{e^x}{x} R(x) \approx \sum_{k=0}^{\infty} R(k) \frac{x^k}{k\ k!} \tag{39}$$

Thus we get $R(x)$ as "entangled" combination of $R(k)$ at positive integers. We end asking the question: Will such a kind of constraint help to prove (3)?

Acknowledgement We thank Prof. L. Baez-Duarte and Prof. K. Maślanka for e-mail exchange. To prepare data for some figures we have used the free package PARI/GP [14].

Appendix

In this appendix we will calculate the error of the approximation $c_k \approx R(k)/k$. Looking at (6) and (28) we see that we have to estimate the sum:

$$\left|\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2}\, e^{-\frac{k}{n^2}} - \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2}\left(1-\frac{1}{n^2}\right)^k\right| < \sum_{n=1}^{\infty}\left|\frac{1}{n^2}\, e^{-\frac{k}{n^2}} - \frac{1}{n^2}\left(1-\frac{1}{n^2}\right)^k\right| \qquad (40)$$

Instead of $\mu(n)$ we have put 1. Let $h(x)$ denote for $1 \leq x$:

$$h(x) = \frac{1}{x^2}\exp(-k/x^2) - \frac{1}{x^2}\left(1-\frac{1}{x^2}\right)^k.$$

This function is bounded by:

$$0 < h(x) \leq \frac{27}{2e^3k^2} + \frac{128}{e^4k^3}.$$

and has one maximum. Thus we can apply the rule:

$$\sum_{n=1}^{\infty} h(n) \leq \max_n h(x) + \int_1^{\infty} h(x)dx.$$

The integral is estimated as

$$\int_1^{\infty} \frac{1}{x^2}\left(\exp(-k/x^2) - (1-1/x^2)^k\right)dx = \int_0^1 (e^{-ky^2} - (1-y^2)^k)dy.$$

$$\int_0^1 e^{-ky^2}dy = k^{-1/2}\int_0^{\sqrt{k}} e^{-y^2}dy \leq k^{-1/2}\int_0^{\infty} e^{-y^2}dy = \frac{\sqrt{\pi}}{2\sqrt{k}}.$$

$$\int_0^1 (1-y^2)^k dy = \frac{4^k}{\binom{2k}{k}(2k+1)} \geq \frac{\sqrt{\pi k}}{2k+1}\left(1+\frac{1}{8k}-\frac{1}{72k^2}\right). \qquad (41)$$

Here the Stirling formula in the form

$$k! = \sqrt{2\pi k}\; k^k e^{-k+\theta(k)}, \qquad \frac{1}{12k+1} < \theta(k) < \frac{1}{12k}$$

was used. Collecting all above estimations we obtain:

$$\int_1^{\infty} \frac{1}{x^2}(\exp(-k/x^2) - (1-1/x^2)^k)dx \leq$$

$$\leq \frac{\sqrt{\pi}}{2\sqrt{k}} - \frac{\sqrt{\pi k}}{2k+1}\left(1+\frac{1}{8k}-\frac{1}{72k^2}\right) < \frac{3\sqrt{\pi}}{16}k^{-3/2} + \frac{\sqrt{\pi}}{144}k^{-5/2}.$$

and finally from the starting sum (40) we get the desired inequality:

$$|R(k)/k - c_k| \leq \frac{3\sqrt{\pi}}{16}k^{-3/2} + \frac{27}{2}e^{-3}k^{-2} + \frac{\sqrt{\pi}}{144}k^{-5/2} + 128e^{-4}k^{-3}.$$

For $k > 16$ it suffices to retain in the above inequality on the r.h.s only the leading term $k^{-3/2}$. Let us remark that the integral (41) can be also taken from tables as it is the Euler Beta integral:

$$\int_0^1 (1-y^2)^k dy = \frac{1}{2}B\left(\frac{1}{2}, k+1\right)$$

and from

$$B(a,x) \sim x^{-a}\Gamma(a) \qquad \text{for } x \text{ large}$$

see e.g. [15]§1.8.7, we get:

$$\int_0^1 (1-y^2)^k dy \sim \frac{\Gamma(\frac{1}{2})}{2\sqrt{k+1}} = \frac{\sqrt{\pi}}{2\sqrt{k+1}}$$

what for large k reproduces leading term in (41).

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